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LETTER TO THE EDITOR

The non-local $\bar{\partial}$ problem and (2+1)-dimensional soliton equations

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Abstract. A method of constructing (2+1)-dimensional non-linear integrable equations and their solutions by means of the non-local $\bar{\partial}$ problem is developed. A 'basic set' of equations is obtained by using different normalisations of the non-local $\bar{\partial}$ problem and the Lagrangian of the set is found. Other integrable equations, which are degenerate cases of the basic set, are also Lagrangian.

A method of constructing (2+1)-dimensional non-linear integrable equations and broad classes of their solutions by means of a non-local Riemann problem (NRP) [1] or a non-local $\bar{\partial}$ problem (NP) [2] (NP is more general, incorporating NRP) was proposed in [3, 4]. This method uses NP with a specific dependence of the kernel on extra variables x_i, $1 \leqslant i \leqslant 3$. One can use it to obtain a class of solutions of some non-linear partial differential equations in x_i, the class depending on a functional parameter, namely the kernel of NP. For kernels sufficiently small in some norm, solutions can be obtained as perturbation series, while for degenerate kernels solutions can be obtained explicitly. In [3, 4], however, the study of integrable equations brought results only in simple special cases and the study of a generic case met with considerable technical difficulties. In this letter we show how these are overcome by choosing *different normalisations* of NP. It had not been noticed earlier that NP admits intrinsically different normalisations, probably because, for the more familiar local Riemann problem used in (1+1) dimensions, all normalisations are gauge equivalent. Thus we emphasise that NP admits such different normalisations in (2+1). Moreover, these allow us to get an integrable set of equations for a generic case in explicit form. We call it the 'basic set' of equations. This set is formally Lagrangian and conserved currents can be obtained for it—the first two are local, but the higher-order currents are non-local. It resembles the three-wave interaction equations in (2+1) but with a special matrix structure, and other integrable equations are degenerate cases of it. In particular, by using the Kadomtsev–Petviashvili (KP) equations as an example, we show that the Lagrangian structure for a degenerate case can be obtained from the Lagrangian of the basic set in some limit. (Comparable results in (1+1) have been obtained by Zakharov and Mikhailov [5].) We discuss the possibilities of finding decreasing solutions with the help of NP and the question of reductions. As a simple and interesting example of the use of different normalisations we treat the Vesselov–Novikov equation. Two different normalisations allow us to introduce a modified Vesselov–Novikov equation as well, and this is a (2+1)-dimensional analogue of the modified Korteweg–de Vries equation (MKdV).

Our approach is based on the non-local $\bar\partial$ problem for complex-valued square, $p\times p$, matrix functions of a complex variable λ. Let $\psi(\lambda)$, $\eta(\lambda)$ be such matrix functions. We refer to the following equation as the non-local $\bar\partial$ problem 'normalised by $\eta(\lambda)$':

$$(\bar\partial+\hat R)\psi=\bar\partial\eta \qquad \psi(\lambda)-\eta(\lambda)\to 0, |\lambda|\to\infty. \tag{1}$$

Here $\bar\partial\equiv\partial/\partial\bar\lambda$ (the bar means complex conjugate) and $\hat R$ is an integral operator acting from the right in the sense of matrix multiplication:

$$(\hat R\psi)(\lambda)=\int\int\psi(\mu)R(\lambda,\mu)\,\mathrm{d}\mu\wedge\mathrm{d}\bar\mu.$$

When $\eta(\lambda)=1$, definition (1) is wholly consistent with the usual definition. We shall consider only rational normalisation functions $\eta(\lambda)$ in this letter.

We introduce the $\bar\partial^{-1}$ operator

$$(\bar\partial^{-1}\phi)(\lambda)\equiv(2\pi\mathrm{i})^{-1}\int\int\phi(\lambda')(\lambda'-\lambda)^{-1}\,\mathrm{d}\lambda'\wedge\mathrm{d}\bar\lambda'$$

so that (1) is

$$\phi+\bar\partial^{-1}\hat R\phi=-\bar\partial^{-1}\hat R\eta \qquad \phi\equiv\psi-\eta\to 0, |\lambda|\to\infty. \tag{2}$$

Equation (2) is a Fredholm integral equation of the second kind in $\phi=\psi-\eta$ (ϕ is the 'regular part' of ψ; it is continuous with $\phi\to 0$, $|\lambda|\to\infty$). When $\hat R$ is small enough in some norm, (2) is solved uniquely for some class of functions. We can therefore assume NP equation (1) is uniquely solvable and therefore that

$$(\bar\partial+\hat R)\psi=0 \qquad \psi(\lambda)\to 0 \qquad |\lambda|\to\infty \qquad \Rightarrow\psi(\lambda)=0. \tag{3}$$

The non-local Riemann problem (NRP) is some limit of NP, (1). Let a continuous oriented curve γ be set on a complex plane. Take a kernel of NP (1) of the form

$$R(\lambda,\mu)=\delta_\gamma(\lambda)R_\gamma(\lambda,\mu)\delta_\gamma(\mu) \tag{4}$$

(where $\delta_\gamma(\lambda)$ is a δ function picking out points on γ). The solution ψ of (1) with kernel (4) is rational outside γ and will have boundary values ψ^+, ψ^- on γ. After regularising $\delta_\gamma(\mu)$ we obtain from NP (1) with kernel (4) the NRP

$$\psi^+-\psi^-=\int_\gamma(\psi^+(\mu)-\psi^-(\mu))R_\gamma(\lambda,\mu)\,|\mathrm{d}\mu|. \tag{5}$$

Evidently NRP normalised by $\eta(\lambda)$ has $\phi=\psi-\eta$ analytic outside γ with $\phi\to 0$, $|\lambda|\to\infty$.

We introduce a dependence of $R(\lambda,\mu)$ (and consequently of ψ) on additional variables x_i, $1\leqslant i\leqslant 3$, through

$$\partial R(\lambda,\mu,\boldsymbol{x})/\partial x_i=K_i(\mu)R-RK_i(\lambda). \tag{6}$$

$K_i(\lambda)$ are commuting matrix-valued functions and $\boldsymbol{x}$ is the set x_i (which, for example, are x, y, t later); the normalisation may depend in an arbitrary way on $\boldsymbol{x}$. A general solution of the three equations (6) is

$$R(\lambda,\mu,\boldsymbol{x})=\exp(K_i(\mu)x_i)R_0(\lambda,\mu)\exp(-K_i(\lambda)x_i) \tag{7}$$

(in which summation over i is understood) while (6) is formally

$$[D_i,\hat R]=0 \tag{8}$$

with $D_i\psi \equiv \partial\psi/\partial x_i + \psi K_i$. From (1) and (8) one has

$$(\bar\partial + \hat R)D_i\psi = [\bar\partial, D_i]\psi + D_i\bar\partial\eta. \tag{9}$$

In this letter we restrict ourselves to the important case of rational $K_i(\lambda)$ for which R defined by (7) has singularities at the poles of $K_i(\lambda)$. We remove these singularities by an appropriate choice of $R_0(\lambda, \mu)$ so that $R(\lambda, \mu, x)$ decreases at λ_p (the pole of K_i) faster than $(\lambda - \lambda_p)^n$, $(\mu - \lambda_p)^n \forall p, x, n$. From (1) it follows that $\bar\partial\phi = \bar\partial(\psi - \eta)$ decreases at λ_p faster than $(\lambda - \lambda_p)^n$ $\forall p, n$. In this case, when $\eta(\lambda)$ is a rational function the solution of NP $D_i\psi$ is also normalised by a rational function.

To prove this note that (9) can be transformed to

$$(\bar\partial + \hat R)D_i\psi = \bar\partial(\eta K_i) + \phi\bar\partial K_i \tag{10}$$

and one can then transform $\phi\bar\partial K_i$ through the formulae

$$\bar\partial(\lambda^{-n}) = \pi(-1)^{n+1}\partial^{n-1}\delta(\lambda)/(n-1)! \tag{11}$$

$$\phi(\lambda)\partial^n\delta(\lambda) = (-1)^n \sum_{k=0}^{n} \delta(\lambda)\binom{n}{k}\partial^k\phi_{|0}\partial^{n-k}. \tag{12}$$

From these it follows that $\phi\bar\partial K_i$ is a sum of δ functions and their derivatives. Then, $\bar\partial^{-1}((\bar\partial\phi)K_i) - \varphi K_i$ is a rational function while the right-hand side of (10) is obviously

$$\bar\partial(\eta K_i + \bar\partial^{-1}((\bar\partial\phi)K_i) - \varphi K_i) \equiv \bar\partial\mu \tag{13}$$

(say) and the solution $D_i\psi$ of NP is normalised by the rational function μ (whose coefficients depend on x). The normalisation μ is a rational part of $D_i\psi$. Indeed it is such a rational function that $\bar\partial(D_i\psi - \mu) = K_i\bar\partial\phi$ is a regular function. Since $\bar\partial\phi$ decreases at the poles of K_i faster than $(\lambda - \lambda_p)^n$ $\forall p, n$, the right-hand side is regular and $D_i\psi - \mu$ is regular.

Thus, the operators D_i make it possible to multiply solutions of NP (1) taken with kernel (7). The number of solutions that can be obtained this way then proves to increase with increasing powers of the operators D_i faster than the dimension of the normalisation divisor (i.e. of the number of poles in μ weighted by their multiplicity) and this induces a linear dependence between solutions which allows us to construct differential equations for the solutions of NP. However, first we derive a 'basic set of equations' by appeal to the statement (3) on the uniqueness of solutions of NP.

So far, rational $K_i(\lambda)$ with poles at one or two points and a single normalisation of NP have been used to construct integrable equations by means of NP. Here we consider the generic case when the $K_i(\lambda)$ each have an arbitrary number n of simple, and distinct, poles:

$$K_i(\lambda) = \sum_{\alpha=1}^{n} a_{\binom{i}{\alpha}}\left[\lambda - \lambda\binom{i}{\alpha}\right]^{-1} \equiv \frac{a_I}{\lambda - \lambda_I} \tag{14}$$

where $I = \binom{i}{\alpha}$, $1 \leqslant \alpha \leqslant n$, is a vector index. The formal expression on the right is labelled by index i and summation over α is understood. The a_I are a set of commuting matrices, while in what follows ijk will be any permutation of indices 123.

We introduce the solutions $\psi_I(x, \lambda)$ of NP (1) normalised by $(\lambda - \lambda_I)^{-1}$. Since $(\hat R\psi)(\lambda)$ decreases as $\lambda \to \lambda_I$ faster than $(\lambda - \lambda_I)^n$ $\forall\psi, I, n$:

$$\begin{aligned}\Psi_I(x, \lambda)_{\lambda\to\lambda_I} &\to (\lambda - \lambda_I)^{-1} + \sum_{n=0}^{\infty} \psi_{II}^{(n)}(x)(\lambda - \lambda_I)^n \\ \Psi_I(x, \lambda)_{\lambda\to\lambda_J} &\to \sum_{n=0}^{\infty} \psi_{IJ}^{(n)}(x)(\lambda - \lambda_J)^n.\end{aligned} \tag{15}$$

The functions $D_i\psi_J$ are also solutions of NP (1). The normalisation of these solutions contains only the first-order poles at the points λ_I, λ_J (and $j\neq i$). One can check by using (15) that $\psi\equiv D_i\psi_J - a_I(\lambda_J-\lambda_I)^{-1}\psi_J - \psi_{JI}^{(0)}a_I\psi_I$ satisfies the conditions of statement (3) on the unique solvability of NP. Hence $D_i\psi_j$, ψ_J and ψ_I are linearly dependent and satisfy (with $i\neq j$)

$$D_i\Psi_J - a_I(\lambda_J-\lambda_I)^{-1}\psi_J - \psi_{JI}^{(0)}a_I\psi_I = 0. \tag{16}$$

(Note that the repeated index I implies summation over α for this i; repeated index J implies no such summation.) The leading order of expansion of (16) as $\lambda\to\lambda_K$ $(i\neq j\neq k)$ yields the equation in the $\psi_{JK}^{(0)}(x)$ (we drop superscript 0)

$$\frac{\partial}{\partial x_i}\psi_{JK} + \psi_{JK}\frac{a_I}{\lambda_K-\lambda_I} = \frac{a_I}{\lambda_J-\lambda_I}\psi_{JK} + \psi_{JI}a_I\psi_{IK}. \tag{17}$$

If the different permutations ijk of the indices are taken into account, (17) is a closed set of $6n^2$ equations: this is what we call the 'basic set' of equations. Normalised solutions of NP give us special solutions of this basic set dependent on the functional parameter $R_0(\lambda,\mu)$. For degenerate $R_0(\lambda,\mu)$ explicit forms of solution can be obtained and these will depend on the behaviour of the exponential functions in (7).

To see this, consider a scalar case. The ψ_I are now complex-valued functions and the a_I are complex numbers. One can obtain solutions $\psi_{IJ}(x)$ satisfying the boundary condition $\psi_{IJ}\to(\lambda_J-\lambda_I)^{-1}$ as $|x|\to\infty$ if the exponents in (7) are imaginary, i.e. $\overline{K_i(\lambda)}-\overline{K_i(\mu)}=K_i(\mu)-K_i(\lambda)$. This condition defines a subset in the space $\mathbb{C}^2$ outside which the kernel of NP $R_0(\lambda,\mu)$ is to be zero. Thus, e.g., when ia_I, λ_J are real it follows that $\mathrm{Im}\,\lambda=0$, $\mathrm{Im}\,\mu=0$ and it is sufficient to use the non-local Riemann problem on the real axis to obtain solutions of (17) satisfying the boundary conditions just given.

The structure of the exponents in (7) also determines the reductions. In terms of the kernel $R(\lambda,\mu,x)$ the simplest reductions are $f(\lambda,\mu)R(\lambda,\mu,x)=R(\lambda_1(\lambda,\mu),\mu_1(\lambda,\mu),x)$ or $f(\lambda,\mu)R(\lambda,\mu,x)=\bar R(\lambda_1(\lambda,\mu),\mu_1(\lambda,\mu),x)$. To satisfy either of these at arbitrary x the exponents in the left- and right-hand parts of these expressions must be equal so that $K_i(\lambda)-K_i(\mu)=K_i(\lambda_1)-K_i(\mu_1)$ or $K_i(\lambda)-K_i(\mu)=\bar K_i(\lambda_1)-\bar K_i(\mu_1)$. These conditions put rigid limitations on the map $\lambda,\mu\to\lambda_1(\lambda,\mu),\mu_1(\lambda,\mu)$.

The basic set of equations (17) is formally Lagrangian with Lagrangian density

$$\begin{aligned}\mathscr{L}(x) = \mathrm{Tr}\{\mathrm{sgn}(ijk)[&\tfrac{1}{2}(\psi_{IJ}a_J\psi_{JI;k}a_I - \psi_{JI}a_I\psi_{IJ;k}a_J)\\ &+a_K[(\lambda_I-\lambda_K)^{-1}\psi_{IJ}a_J\psi_{JI}a_I - (\lambda_J-\lambda_K)^{-1}\psi_{JI}a_I\psi_{IJ}a_J]\\ &+\tfrac{1}{3}(a_I\psi_{IK}a_K\psi_{KJ}a_J\psi_{JI} - a_I\psi_{IJ}a_J\psi_{JK}a_K\psi_{KI})]\}\end{aligned} \tag{18}$$

where $\mathrm{sgn}(ijk)$ is the sign of the permutation (ijk), and summation is over α in common indices I as well as over permutations of the indices ijk. The Lagrangian (18) was constructed by analogy with the three-wave interaction equations.

Expansion of (16) about λ_J gives a set of non-local (in general) conserved currents: only the first and second currents are local. Here we confine attention to these two currents, but it is easy to obtain recursion relations for the whole set by the same arguments. Expansion of (16) about λ_J to leading order is $(\psi_{JJ;i}\equiv\partial\psi_{JJ}/\partial x_i)$

$$\psi_{JJ;i} + [\psi_{JJ}, a_I(\lambda_J-\lambda_I)^{-1}] - \frac{a_I}{(\lambda_J-\lambda_I)^2} = \psi_{JI}a_I\psi_{IJ}. \tag{19}$$

J is not summed but I is. By differentiating (19) with respect to x_k and the same equation, but with i changed to k, with respect to x_i we obtain the first conserved current as

$$\frac{\partial}{\partial x_k}\mathrm{Tr}(A\psi_{JI}a_I\psi_{IJ})=\frac{\partial}{\partial x_i}\mathrm{Tr}(A\psi_{JK}a_K\psi_{KJ})$$
$$[A, a_I]=[A, a_K]=0 \qquad \forall I, K. \tag{20}$$

The next order of expansion of (16) as $\lambda\to\lambda_J$ yields

$$\psi_{JJ}^{(1)}+[\psi_{JJ}^{(1)}, a_I(\lambda_J-\lambda_I)^{-1}]-\psi_{JJ}^{(0)}a_I(\lambda_J-\lambda_I)^{-2}+2a_I(\lambda_J-\lambda_I)^{-3}=\psi_{JI}a_I\psi_{IJ}^{(1)}. \tag{21}$$

To eliminate the $\psi_{IJ}^{(1)}(x)$ expand (16), with i, j, interchanged, about λ_J to leading order. Then

$$\psi_{IJ}^{(1)}a_J=\psi_{IJ'}a_{J'}\psi_{J'J}+a_{J'}(\lambda_I-\lambda_{J'})^{-1}\psi_{IJ}-\psi_{IJ}a_{J''}(\lambda_J-\lambda_{J''})^{-1}-\psi_{IJ;j} \tag{22}$$

and summation is over J' and J'' ($J''\neq J$). By choosing A and using (22) we obtain, as for (19), that the second current satisfies

$$\frac{\partial}{\partial x_k}\mathrm{Tr}[a_J\psi_{JI}a_Ia_{J'}\psi_{IJ}(\lambda_I-\lambda_{J'})^{-1}-a_J\psi_{JI}a_I\psi_{IJ}a_{J''}(\lambda_J-\lambda_{J''})^{-1}-a_J\psi_{JI}a_I\psi_{IJ;j}]$$
$$=\frac{\partial}{\partial x_i}\mathrm{Tr}[a_J\psi_{JK}a_Ka_{J'}\psi_{KJ}(\lambda_K-\lambda_{J'})^{-1}$$
$$-a_J\psi_{JK}a_K\psi_{KJ}a_{J''}(\lambda_J-\lambda_{J''})^{-1}-a_J\psi_{JK}a_K\psi_{KJ;j}] \tag{23}$$

where summation is over I, K, J', J'', J, ($J''\neq J$). Notice that both the basic set of equations (17) and the conserved currents for them are obtained from (16), i.e. equations (16) contain complete information about both the equations of motion and the conserved currents.

Earlier we noted that the operators D_i allow us to multiply solutions of NP (1) and when there arises a linear dependence between such solutions it is possible to obtain differential equations for the normalised solutions of NP. In fact, it proves that, if some linear combination of solutions to NP obtained through the D_i is equal to zero, it can be represented as a linear combination of derivatives D_i acting on the expression on the left-hand side of (16). This allows us to construct integrable equations with higher-order derivatives.

Let us consider solutions of NP of the form

$$P(x,\lambda)=\sum_{p,q,r=0}^{N}u_I(p,q,r,x)D_i^pD_j^qD_k^r\psi_I(x,\lambda)+\mathrm{CP}$$

where CP means cyclic permutations of i, j, k. Such expressions will be identical if all coefficients $u_I(p, q, r, x)$ are equal. Define (for $i\neq j$)

$$P_{iJ}(x,\lambda)\equiv D_i\psi_J-a_I(\lambda_J-\lambda_I)^{-1}\psi_J-\psi_{JI}a_I\psi_I. \tag{24}$$

Then (16) means that $P_{iJ}=0$. We prove the theorem

$$P(x,\lambda)=0\Rightarrow P(x,\lambda)\equiv\sum_{p,q,r=0}^{N}u_{iJ}(p,q,r,x)D_i^pD_j^qD_k^rP_{iJ}+\mathrm{CP}. \tag{25}$$

We use two lemmas.

Lemma 1.

$$D_i^p D_j^q D_k^r \psi_I(x, \lambda) \equiv \sum_{n=0}^{N} u_I(n, x) D_i^n \psi_I(x, \lambda) + \sum_{p,q,r=0}^{N} u_{iJ}(p, q, r, x) D_i^p D_j^q D_k^p P_{iJ}(x, \lambda) + \text{CP}. \tag{26}$$

Summations are over the same indices I or J with α running from l to n in each case. The proof is by induction: formula (24) means all expressions of the form $D_i\psi_J$ can be eliminated from the left-hand side.

Lemma 2.

$$\sum_{n=0}^{N} u_I(n, x) D_i^n \psi_I(x, \lambda) + \text{CP} = 0 \Rightarrow \sum_{n=0}^{N} u_I(n, x) D_i^n \psi_I(x, \lambda) \equiv 0.$$

Only $D_i^N \psi_I(x, \lambda)$ has at λ_I a pole of order $N+1$. This implies that the coefficient $u_I(N, x) \equiv 0$. The lemma then follows by induction.

Evidently the two lemmas immediately imply the theorem stated in (24). The theorem allows us to construct integrable equations with higher-order derivatives: details will be given elsewhere.

To integrate (i.e. find solutions of) the basic set of equations we have used $K_i(\lambda)$ of the form (14). By 'degenerating' these one can obtain arbitrary rational $K_i(\lambda)$ in appropriate limits. Taking KP as an example we show that the Lagrangian for this degenerate case can be obtained from Lagrangian (17) of the basic set in such a limit.

To integrate KP choose $D_1 = \partial/\partial x + \lambda^{-1}$, $D_2 = \partial/\partial y + \lambda^{-2}$, $D_3 = \partial/\partial t + \lambda^{-3}$. Let $\psi_1(x, y, t, \lambda)$ be normalised by λ^{-1}. Then, after some work, one can show from (17) that $u = \psi_{11}(x, y, t)$ satisfies

$$\frac{\partial}{\partial x}(u_t - \tfrac{1}{4}u_{xxx} + \tfrac{3}{2}u_x u_x) = \tfrac{3}{4}u_{yy} \tag{27}$$

which is the KP equation in the usual form for $v = u_x$. Here we calculate the Lagrangian. Multiple poles of $K_i(\lambda)$ can be obtained as a limit from simple poles. Consider D_i' of the form

$$D_1' = \partial/\partial x + \lambda^{-1} \qquad D_2' = \partial/\partial y + (2\varepsilon)^{-1}[(\lambda - \varepsilon)^{-1} - (\lambda + \varepsilon)^{-1}]$$
$$D_3' = \partial/\partial t + (2\varepsilon^2)^{-1}[(\lambda + \varepsilon)^{-1} + (\lambda - \varepsilon)^{-1} - 2\lambda^{-1}].$$

For $\varepsilon \to 0$ these three D_i integrate KP but as defined they have coinciding poles. To remove these make the linear transformation of coordinates

$$D_1' = D_1 \qquad D_2' = \tfrac{1}{2}(\varepsilon^2 D_3 + \varepsilon D_2 + D_1) = \partial/\partial p + (\lambda - \varepsilon)^{-1}$$
$$D_3' = \tfrac{1}{2}(\varepsilon^2 D_3 - \varepsilon D_2 + D_1) = \partial/\partial q + (\lambda + \varepsilon)^{-1}. \tag{28}$$

The functions $K_i'(\lambda)$ now have simple poles each at different points $0, \varepsilon, -\varepsilon$. Denote these by indices 1, 2, 3, respectively. The Lagrangian (18) for the case (28) is

$$\mathcal{L}(x, y, t) = \psi_{12}\frac{\partial}{\partial q}\psi_{21} + \psi_{23}\frac{\partial}{\partial x}\psi_{32} + \psi_{31}\frac{\partial}{\partial p}\psi_{13} + \frac{1}{2\varepsilon}\psi_{12}\psi_{21} + \frac{1}{2\varepsilon}\psi_{13}\psi_{31} + \frac{2}{\varepsilon}\psi_{23}\psi_{32} + \psi_{13}\psi_{32}\psi_{21} - \psi_{12}\psi_{23}\psi_{31}. \tag{29}$$

To obtain the Lagrangian of KP from (29) in the limit $\varepsilon \to 0$ it is necessary to express all functions in (29) in terms of the expansion of $\psi_1(x, y, t, \lambda)$ as $\lambda \to 0$ by using the the basic set of equations (17) for the case (28). Two of these, namely

$$\psi_{13;p} + \frac{1}{2\varepsilon}\psi_{13} - \psi_{12}\psi_{23} = 0 \tag{30}$$

$$\psi_{12;q} - \frac{1}{2\varepsilon}\psi_{12} + \psi_{13}\psi_{32} = 0 \tag{31}$$

make it possible to exclude ψ_{23}, ψ_{32} from the Lagrangian and put it in the form (omitting any total derivatives)

$$\mathscr{L}(x, y, t) = (\psi_{12})^{-1}\left(\psi_{13;p} + \frac{1}{2\varepsilon}\psi_{13}\right)\frac{\partial}{\partial x}\left[(\psi_{13})^{-1}\left(\psi_{12;b} - \frac{1}{2\varepsilon}\psi_{12}\right)\right]$$
$$+ 2\varepsilon^{-1}(\psi_{13;p}\psi_{12;q}/\psi_{12}\psi_{13}). \tag{32}$$

Expressions (30)-(32) can be rewritten in terms of functions with subtracted singularities $\phi_{ij} = \psi_{ij} - (\lambda_j - \lambda_i)^{-1}$ as

$$\phi_{23}(1 + \varepsilon\phi_{12}) = \tfrac{1}{2}(\phi_{12} + \phi_{13}) + \varepsilon\phi_{13;p} \tag{33}$$

$$\phi_{32}(-1 + \varepsilon\phi_{13}) = -\tfrac{1}{2}(\phi_{12} + \phi_{13}) + \varepsilon\phi_{12;q} \tag{34}$$

$$\mathscr{L}(x, y, t) = 2\varepsilon^{-1}[\partial \ln(1 + \varepsilon\phi_{12})/\partial q][\partial \ln(1 - \varepsilon\phi_{13})/\partial p]$$
$$+ (1 + \varepsilon\phi_{12})^{-1}[\tfrac{1}{2}(\phi_{12} + \phi_{13}) + \varepsilon\phi_{13;p}]$$
$$\times \partial\{(-1 + \varepsilon\phi_{13})^{-1}[-\tfrac{1}{2}(\phi_{12} + \phi_{13}) + \varepsilon\phi_{12;q}]\}/\partial x \tag{35}$$

respectively. If one writes the expansion of ϕ_{12} and ϕ_{13} in powers of ε as

$$\phi_{12} = u + \phi_{11}^{(1)}\varepsilon + \phi_{11}^{(2)}\varepsilon^2 + \phi_{11}^{(3)}\varepsilon^3 + \mathrm{O}(\varepsilon^3)$$

$$\phi_{13} = u - \phi_{11}^{(1)}\varepsilon + \phi_{11}^{(2)}\varepsilon^2 - \phi_{11}^{(3)}\varepsilon^3 + \mathrm{O}(\varepsilon^3)$$

the expansion of the Lagrangian (35) is

$$\mathscr{L}(x, y, t) = 2\varepsilon^3(-u_t u_x - \tfrac{1}{4}u_{xx}^2 - u_x^3 + \tfrac{3}{4}u_y^2) + \mathrm{o}(\varepsilon^3). \tag{36}$$

To eliminate $\phi_{11}^{(1)}$ one uses $\partial\phi_{11}^{(1)}/\partial x = \frac{1}{2}u_y + uu_x - \frac{1}{2}u_{xx}$ which arises at zero order in ε directly from the basic set of equations. The expression in parentheses in (36) is a well known Lagrangian for the KP equation (27). Thus we have obtained the Lagrangian of KP as a limit from the Lagrangian (18) of the basic set of equations.

The Vesselov-Novikov equation is a simple and interesting example of the use of different normalisations. It is one of the (2+1)-dimensional analogues of the KdV equation in (1+1) and its Lax L operator is a true two-dimensional Schrödinger operator [6]. It has been integrated using one normalisation of NP. But the use of the two normalisations natural to this problem allows us to construct a set of two Lagrangian equations, reductions of which are the Vesselov-Novikov equation and a second equation we call the modified Vesselov-Novikov equation, as well as the Miura transformation between the solutions of these two equations [7, 8]. In this way the analogy with the (1+1)-dimensional case in which KdV and MKdV are obtained as reductions from a pair of equations is made both clear and complete.

Consider

$$D_1 = \partial/\partial z + \mathrm{i}\lambda^{-1} \qquad D_2 = \partial/\partial\bar{z} + \mathrm{i}\lambda \qquad D_3 = \partial/\partial t + \mathrm{i}(\lambda^3 + \lambda^{-3}). \tag{37}$$

The solutions of NP $\psi_1(\lambda, z, t)$ and $\psi_2(\lambda, z, t)$ are normalised by λ^{-1} and 1, respectively. One then finds $f=\psi_{12}(z, t)$ and $g=\psi_{21}(z, t)$ satisfy the pair of coupled equations, with boundary condition $f, g \to l$ as $|z| \to \infty$,

$$(\partial_t+\partial^3+\bar{\partial}^3)f+3(\partial f)\bar{\partial}^{-1}\partial(fg)+3(\bar{\partial}f)\partial^{-1}\bar{\partial}(fg)+3f\partial^{-1}\bar{\partial}(g\bar{\partial}f)+3f\bar{\partial}^{-1}\partial(g\partial f)=0 \qquad (38a)$$

$$(\partial_t+\partial^3+\bar{\partial}^3)g+3(\bar{\partial}g)\partial^{-1}\bar{\partial}(fg)+3(\partial g)\bar{\partial}^{-1}\partial(fg)+3g\bar{\partial}^{-1}\partial(f\partial g)+3g\partial^{-1}\bar{\partial}(f\bar{\partial}g)=0 \qquad (38b)$$

with $\partial \equiv \partial/\partial z$, $\bar{\partial} \equiv \partial/\partial \bar{z}$, $\partial_t \equiv \partial/\partial t$. The Lagrangian of this pair is

$$\mathscr{L}(z, t)=\mathrm{i}\{g(\partial_t+\partial^3+\bar{\partial}^3)f+3(fg-1)[\partial^{-1}\bar{\partial}(g\bar{\partial}f)+\bar{\partial}^{-1}\partial(g\partial f)]-\mathrm{cc}\} \qquad (39)$$

in which CC is complex conjugate. The reduction of (38) by $g=1$ yields the Vesselov-Novikov equation

$$(\partial_t+\partial^3+\bar{\partial}^3)f+3\partial(f\bar{\partial}^{-1}\partial f)+3\bar{\partial}(f\partial^{-1}\bar{\partial}f)=0. \qquad (40)$$

On the other hand, the reduction $f=\bar{g}$ gives the modified Vesselov-Novikov equation

$$(\partial_t+\partial^3+\bar{\partial}^3)f+3(\partial f)\bar{\partial}^{-1}\partial(\bar{f}f)+3(\bar{\partial}f)(\partial^{-1}\bar{\partial}(\bar{f}f))+3f\partial^{-1}\bar{\partial}(\bar{f}\bar{\partial}f)+3f\bar{\partial}^{-1}\partial(\bar{f}\partial f)=0 \qquad (41)$$

The Lagrangian of the modified Vesselov-Novikov equation (41) is obtained from (39) by the reduction $f=\bar{g}$. For the Vesselov-Novikov equation (40) the Lagrangian (39) under the reduction $g=1$ is equal to zero.

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